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Orthotropic index for bone

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Abstract An orthotropic index (*OI*) is proposed to indicate the existence of a preferred material direction in each of the symmetry planes of an orthotropic material such as bone. Currently, this function is performed by the anisotropy ratio (*AR*) of any two Young's moduli or compressive (A_c) and shear (A_s) anisotropy factors comprised of complicated functions of the elastic constants. The *OI* incorporates the four independent engineering constants (the shear modulus and Poisson's ratio in addition to the two Young's moduli) in each symmetry plane into a single index. The *OI* thus improves upon the *AR* by reflecting orthotropy in a more holistic sense and upon the *AR*, A_c and A_s by taking on a unique value (zero) only when the material is in fact isotropic.

1. Introduction

Bone usually is mechanically modeled as a linear elastic orthotropic material. Interpretation and comparison of measured elastic constants is obscured by the number needed to characterize an orthotropic material: nine, including three Young's moduli, three shear moduli, and three Poisson's ratios. One interpretation, the anisotropy ratio (*AR*) of any two principal Young's moduli [8], indicates the existence of preferred material directions but incorporates only two constants. Another interpretation is represented by the compressive (A_c) and shear (A_s) anisotropy factors [2,3,4], which incorporate all nine elastic constants into these two factors. The objective of this work is to propose an index that enables comparisons between the constants in a more inclusive sense. The four elastic constants in each of the material symmetry planes are collapsed into a single orthotropic index (*OI*) that indicates the degree of orthotropy in each of the planes. This *OI* is demonstrated to take on a unique value for isotropic materials (the *AR*, A_c and A_s do not) and to retain the desirable characteristics of the *AR*, i.e., it is easy to apply and increases with the degree of material orientation.

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2. Methods

Plane stress problems without body forces in orthotropic elasticity theory reduce to determining the stress function $F(x_1, x_2)$ which satisfies

$$\frac{E_{11}}{E_{22}}\frac{\partial^4 F}{\partial x_1^4}+\left(\frac{E_{11}}{G_{12}}-2\nu_{12}\right)\frac{\partial^4 F}{\partial x_1^2\partial x_2^2}+\frac{\partial^4 F}{\partial x_2^4}=0 \quad (1)$$

and boundary conditions [6]. The E_{11} and E_{22} are Young's moduli, ν_{12} a Poisson ratio, G_{12} a shear modulus. The subscripts "1" and "2" denote principal material directions in the x_1-x_2 symmetry plane. The roots μ_1 and μ_2 of the characteristic equation of Eq. (1) are such that

$$-i(\mu_1+\mu_2)=\sqrt{2\left(\frac{E_{11}}{E_{12}}-\nu_{12}\right)+\frac{E_{11}}{G_{12}}}\equiv 2\eta_{12} \quad (2)$$

where $i=\sqrt{-1}$. The center of the equality of Eq. (2) contains the four constants which describe elastic behavior in the x_1-x_2 plane and provides a ready nondimensional form from which the definition of the *OI* is based

$$OI_{12} \equiv \max(|\eta_{12} - 1|, |\eta_{21} - 1|) \quad (3)$$

where η_{21} is defined similarly by indicial substitution in Eq. (2). Similarly, OI_{23} and OI_{13} can be formulated as well.

3. Results and discussion

The *OI* may be a more useful indicator of orthotropy if it improves upon the *AR* while retaining its desirable characteristics. One improvement is the incorporation of twice as many elastic constants. Another improvement is that the *OI* takes on a unique value for isotropic materials whereas the *AR* does not. Using the x_1-x_2 symmetry plane as an example, $AR_{12} = 1$ for materials that are isotropic in that plane. Since $E_{11} = E_{22}$ and $G_{12} = 1/2E_{11}/(1 + \nu_{12})$ for isotropic materials, it can be shown from Eqs. (2) and (3) that $\eta_{12} = \eta_{21} = 1$, so that $OI_{12} = 0$ is then true. However, an infinite number of possible orthotropic materials may still have $AR_{12} = 1$; for example, $E_{11} = E_{22} = 20$ GPa, $G_{12} = 4$ GPa, and $\nu_{12} = 0.4$ is one such orthotropic material. Orthotropic materials are such that $OI_{12} \neq 0$ will always be true, as the *OI* is defined herein. This is graphically demonstrated by plotting (Fig. 1) "regions" of the ratio of the primary Young's modulus to the shear modulus, E_{11}/G_{12}, so that $\eta_{12} = 1$, i.e.,

$$\frac{E_{11}}{G_{12}} = 2\left(2 - \frac{E_{11}}{E_{22}} + \nu_{12}\right) \quad (4)$$

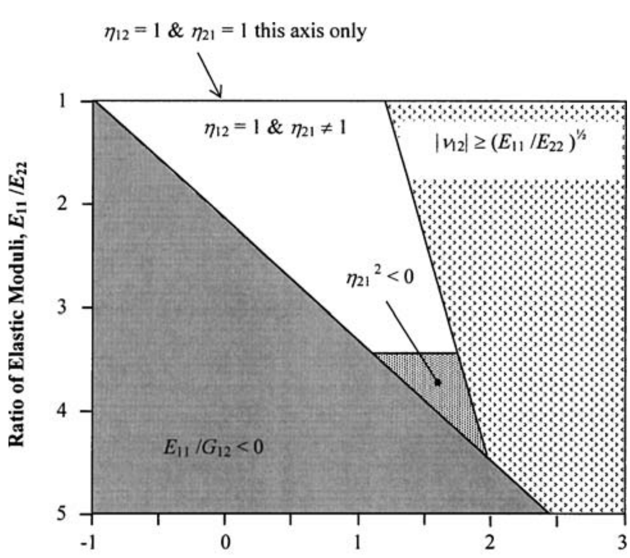

Fig. 1 "Regions" of the ratio of the primary Young's modulus to the shear modulus, E_{11}/G_{12}, as a function of the ratio of the Young's moduli E_{11}/E_{22} and the Poisson's ratio ν_{12} so that $\eta_{12} = 1$ for orthotropic materials.

as a function of the ratio of the Young's moduli E_{11}/E_{22} and the Poisson's ratio ν_{12} for a variety of (possible and impossible) orthotropic materials. Note that the ordinate scale in Fig. 1 is increasing downward. The three shaded regions represent impossible orthotropic materials that violate thermodynamic constraints (shown in the shaded areas) on the engineering constants [7]. Possible orthotropic materials exist in the unshaded central region, such that $\eta_{12} = 1$ and $\eta_{21} \neq 1$, making $OI_{12} \neq 0$. Only along the upper axis lie isotropic materials for which $OI_{12} = 0$. Similar statements can be made regarding the other symmetry planes. This plot demonstrates that the *OI* defined herein takes on a unique value, zero, for isotropic materials only.

The desirable characteristics of the *AR* are maintained by the *OI*. One such characteristic is that the *OI* is easy to apply through a simple calculation, although admittedly less easy to memorize than the *AR*. Another such characteristic is that for bone and common engineering materials (in which $AR_{12} = E_{11}/E_{22} > 1$ if x_1 and x_2 are so chosen, $E_{11}/G_{12} > 1$, and $0.2 < \nu_{12} < 0.4$ are generally true), the *OI* can be shown to be a monotonically increasing function of E_{11}/E_{22}. Thus higher degrees of orthotropy as indicated by increasing *AR*s are reflected by increasing *OI*s as well.

The *OI* defined herein may prove to be a useful descriptor of bone orthotropic elasticity. Engineering constants, determined from ultrasonic measurements of bovine Haversian and plexiform [5] and human Haversian [1] bone were used to compute various *OI*s and compare them to their respective *AR*s (Fig. 2). The nearer an *OI* is to zero, the more the bone

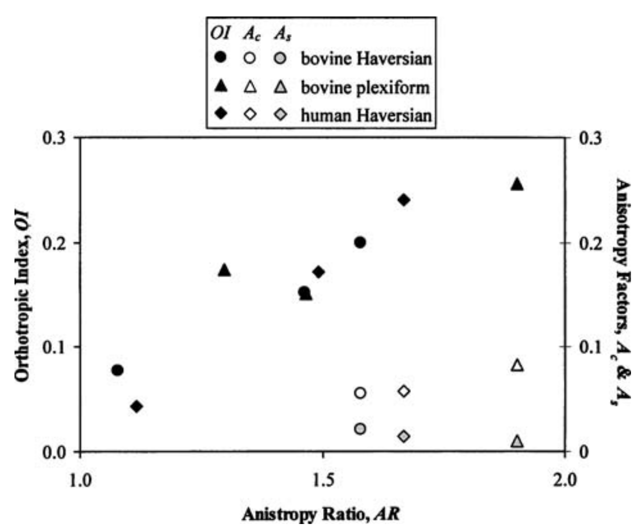

Fig. 2 Comparison between the orthotropy factors (*OI* denoted by black-filled symbols) proposed herein and the anisotropy factors (A_c denoted by unfilled symbols and A_s by gray-filled symbols) of [2, 3, 4] versus anisotropy ratios (*AR*) for different bone types (bovine Haversian denoted by circles, bovine plexiform by triangles and human Haversian by diamonds). Isotropic materials plot at the coordinates (*AR, OI*) = (1, 0), (*AR*, A_c) = (1, 0) and (*AR*, A_s) = (1, 0), but not all materials for which $AR = 1$, $A_c = 0$ or $A_s = 0$ are isotropic.

behaves like an isotropic material in that plane. Bone with a highly preferred material direction will be indicated by a relatively large *OI*, as is the case with the *AR*. However, *OI*s can be more discriminating than the *AR*s: differences between *OI*s for all 3 combinations of bone types (Fig. 2) taken two at a time were virtually always greater than differences between the *AR*s. By including more elastic constants in this type of descriptor, subtle differences in orthotropy become apparent, as opposed to the simply constructed *AR*. Further, for the data shown in Fig. 2, a clear increasing trend in A_c or A_s with increasing *AR* is not evident. Furthermore, the *AR*s, A_c and A_s can give false positives for isotropy, while the *OI*s cannot: admissible orthotropic (and not isotropic) materials can be found that result in $AR = 1$ or $A_c = 0$ and $A_s = 0$. Finally, the *OI* has an important mechanical interpretation, being related to stress concentrations about holes in orthotropic plates [6].

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